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A Three-Level Mathematical Programming Model of Road Pricing

PAOLO FERRARI
School of Engineering of Pisa, Via Diotisalvi 2, 56126 Pisa, Italy

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Abstract. The paper presents a model of toll imposition on road networks, with the aim of recovering, at least in part, the road costs. This model supposes that the amount of the toll is the result of negotiations between the public agency, which is in charge of the network management, and tends to maximize the toll revenues, and the local communities, who bear the negative consequences of the toll imposition, and are compensated for the damage they sustain. A method of negotiation management is proposed, which compels both parties to behave correctly. The model leads to a three-level hierarchical optimization problem for which a method of solution is proposed and, by way of example, applied to study case.

Key words: Compensation, Hierarchical optimization, Road network, Road pricing

1. Introduction

For some time now the public agencies of various countries have been studying the advisability of levying tolls on some roads of their networks, typically the motorways and other roads with characteristics similar to motorways, with the aim to recoup, at least in part, the costs of road improvement, maintenance and management. The toll imposition provokes the transfer of part of traffic from roads on which tolls are imposed to alternative roads, which often traverse areas that are either inhabited or particularly important from the environmental point of view, and thus where increased traffic may have a particularly strong impact [2]. This impact gives rise to conflicts between the public agency which is in charge of the network management and whose financial needs call for application of the toll, and the local communities whose territories must bear the consequences of the increased traffic on the routes alternative to the toll roads.

One way that these conflicts could be resolved is to appraise the environmental damage sustained by local communities and turn over a share of the toll revenues to them as compensation. Fixing the exact amount of such compensation would be a matter for negotiations between the public agency and the local communities, negotiations that should be conducted following rules able to ensure correct behavior from both parties.

This paper presents a model of negotiations, whose rules assign to the public agency the task of defining a compensation function, which provides the monetary value of the compensation as a function of the toll. The amount of the toll itself is instead set by the local communities so as to maximize the profit accruing to them from the compensation, that is, the difference between the compensation and the monetary value that they attribute to the damage they sustain.

This model of negotiation management leads to a three-level hierarchical game: at the upper level the public agency defines the compensation function; at the middle level the local communities choose the toll on the basis of the established compensation function; and at the lowest level, the network users, given the toll, distribute themselves amongst the various routes and thereby determine the degree of damage caused to the local communities.

2. The Model

Let us consider a road network represented by a graph *G(N,L)*, in which *N* is the set of nodes and *L* the set of links: some nodes are centroids of origin and destination of transportation demand. We study the network in a time interval τ during which the transportation demand is constant, and τ is long enough for traffic flow to be considered stationary during it.

A toll is going to be levied on road *A* of the network, which is composed of a subset *I* of the links of the graph. Let *W* be the set of pairs, *w*, of origin-destination centroids; d_w, the transportation demand between pair *w* during one hour of the interval τ; π, the toll imposed per km on the links $i \in I$; l_i, the length in km of link $i \in L$; f_i, the hourly vehicular flow on link $i \in L$; *f*, the vector of the flows $f_i, \forall i \in L$; and Θ, the set of vectors, *f*, feasible on the network.

We assume that the network capacity constraints are not active. If the network is studied via a deterministic approach, and the costs functions $c_i(f_i)$ on links $i \in L$ are separable, then the hourly flow vector at equilibrium on the network is the solution to the following minimization problem:

$$\min_{f \in \Theta} F_1(f) = \sum_{i \notin I} \int_0^{f_i} c_i(x)\mathrm{d}x + \sum_{i \in I} \int_0^{f_i} (c_i(x) + \pi \cdot l_i)\mathrm{d}x \qquad (2.1)$$

Adopting, instead, a stochastic approach, let $C_w(f)$ be the mean of the minimum travel costs incurred by motorists between $w \in W$ when the average flow vector is *f*, and let $c_i(f_i)$ be the mean of the costs incurred by motorists to travel link $i \in L$, which is a function of the flow f_i. In this case, the vector of the mean equilibrium flows during interval τ is the solution to the following minimization problem [4]:

$$\min_{f\in\Theta} F_2(f) = -\sum_{w\in W} C_w(f)d_w + \sum_{i\notin I}\left[f_i c_i(f_i) - \int_0^{f_i} c_i(x)\mathrm{d}x \right] +$$
$$+ \sum_{i\in I}\left[f_i(c_i(f_i)+\pi\cdot l_i) - \int_0^{f_i}(c_i(x)+\pi\cdot l_i)\mathrm{d}x \right] \qquad (2.2)$$

Since both problems (1) and (2) have a unique solution, both implicitly define the equilibrium hourly flow vector f as a function of $\pi: f = f(\pi)$.

Now let $\beta g(f)$ be the monetary value that the local communities attribute to the damage they sustain during one hour of interval τ. We suppose that $g(f)$ is a known scalar function of equilibrium flow vector f, which can be calculated as a result of the solution to one of the two problems (1) or (2). For example, $g(f)$ may be a function of the sum of the increases in travel time of a vehicle on the links belonging to routes alternative to road A consequent to institution of the toll; since f is function of π, even g is function of $\pi: g = g(f(\pi))$. β is a parameter known to the local communities, and unknown to the public agency.

Let $\xi(\alpha,\pi)$ be the monetary compensation paid by the public agency to the local communities for every hour of the interval τ. It is assumed that ξ is a quadratic concave function of the toll π:

$$\xi(\alpha,\pi) = \alpha_1\pi - \alpha_2\pi^2 \qquad (2.3)$$

where α_1 and α_2 are the components of a parameter vector α defined by the public agency.

Given the compensation function $\xi(\alpha,\pi)$ the toll π is chosen by the local communities so as to maximize the profit U accruing from the compensation, that is, the difference between the compensation and the damage $\beta g(f(\pi))$, under the constraints that neither profit nor toll are negative. Thus π is solution of the following problem:

$$\max_\pi U = \max_\pi[\xi(\alpha,\pi) - \beta g(f(\pi))] \;\; s.t. \;\; U \geqslant 0, \;\; \pi \geqslant 0 \qquad (2.4)$$

If the public agency knew the value attributed to parameter β by the local communities, it would define the parameter vector α so as to maximize the net toll revenues, that is, the difference between the toll revenues, $R(f(\pi),\pi) = \sum_{i\in I} f_i(\pi)\cdot l_i\cdot\pi$ and the compensation $\xi(\alpha,\pi)$.

Thus the model would give rise to a three-level optimization problem [1], where the public agency would solve the problem:

$$\max_\alpha[R(f(\pi(\alpha,\beta)),\pi(\alpha,\beta)) - \xi(\alpha,\pi(\alpha,\beta))] \qquad (2.5)$$

where function $\pi(\alpha\beta)$ is implicitly defined as solution of the problem (2.4) faced by the local communities:

$$\max_\pi[\xi(\alpha,\pi) - \beta g(f(\pi))] \;\; s.t. \;\; U \geqslant 0, \;\; \pi \geqslant 0 \qquad (2.6)$$

and $g(f(\pi))$ is implicitly defined by one of two problems (2.1) or (2.2) depending, on whether the demand is assigned to the network via a deterministic procedure or a stochastic one.

In real networks, as we will see later, $g(f(\pi))$ is, with very good approximation, a strictly convex quadratic function of π:

$$g(f(\pi)) = y(\pi) = a\pi + b\pi^2 \tag{2.7}$$

and $R(f(\pi))$ is a strictly concave quadratic function of π:

$$R(f(\pi)) = S(\pi) = c\pi - d\pi^2 \tag{2.8}$$

By substituting expressions (2.3) and (2.6) into problem (2.4), the latter is written:

$$\max_{\pi}((\alpha_1 - a\beta)\pi - (\alpha_2 + b\beta)\pi^2) \tag{2.9}$$

and its solution gives raise to the following expression of the value chosen by the public communities as a function of α and β:

$$\pi(\alpha,\beta) = \frac{\alpha_1 - a\beta}{2(\alpha_2 + b\beta)} \quad \text{if } \ \pi(\alpha,\beta) \geqslant 0\ , \text{ otherwise } \pi = 0 \tag{2.10}$$

On the other hand, by substituting expressions (2.3), (2.8) and (2.10) into problem (2.5), the latter is written:

$$\max_{\alpha}\left[(c-\alpha_1)\frac{\alpha_1 - a\beta}{2(\alpha_2 + b\beta)} - (d-\alpha_2)\frac{(\alpha_1 - a\beta)^2}{4(\alpha_2 + b\beta)^2}\right] \tag{2.11}$$

By solving problem (2.11) the public agency would thoroughly define the compensation function, while function (2.10) would furnish the toll chosen by the local communities for a given monetary value β attributed by them to the damage they sustain.

But the public agency does not know the value that the local communities attribute to β, and can only make a guess on it. If this guess is different from the value chosen by the local communities, the values of parameters α_1 and α_2 computed by the public agency would be different from those which would be obtained from problem (2.11) by using the β value chosen by the local communities; correspondingly different, and less, would be the net toll revenues. From expression (2.11) we deduce that the difference is small if parameters a and b of the damage function $g(f(\pi))$ are low, which means that the damage caused by the toll imposition is scarce. But if a and b are high, the reduction in toll revenues consequent to a guess on β quite different from the value attributed to it by the local communities could be substantial.

For these reasons it is important that the public agency admits its ignorance, and considers β as a random variable distributed around a mean μ_β with variance σ_β^2.

In this case the problem it has to solve is the maximization of the expectation of the net toll revenues:

$$\max_{\alpha}\int_{\beta_1}^{\beta_2}\left[R(f(\pi(\alpha,\beta)),\pi(\alpha,\beta))-\xi(\alpha,\pi(\alpha,\beta))\right]f_{\beta}(\beta)\mathrm{d}\beta \tag{2.12}$$

where β_1 and β_2 are the extremes of the interval in which β is defined, and $f_\beta(\beta)$ is the probability density function of β. Thus problem (2.12) substitutes problem (2.5) in the hierarchical optimization.

By the exam of function (2.12) we observe that, if the public agency has scarce information about the evaluation of the damage due to toll imposition on the part of local communities, so that there is high probability that the β value the latter chooses is rather different from μ_β, it is important that the public agency attributes high value to σ_β^2, so that the weights of the net toll revenues in function (2.12) are rather high even in correspondence of β values quite different from μ_β.

We observe that the method of conducting the negotiations proposed in this paper prevents both parties from having improper behaviors. In fact it would penalize the public agency if, in calculating the compensation function, it assumed a probability law for the damage monetary value that did not reflect its true awareness of the local communities estimate for this value. On the other hand the local communities would be penalized if, in the computation of the toll which maximizes the profit accruing from the compensation, they would use a damage monetary value greater than that they really estimate: in fact, any toll value computed by solving problem (2.4) with a β value different from the real one would lead to lower profit. Thus we can say that the proposed method of conducting the negotiations works as *revelation mechanism* [3] of the monetary value that the local communities assign to the damage sustained.

3. A Method of Solution

As we have seen, the solution of the three-level optimization problem proposed in Section 2 calls for the computation of functions $y(\pi)=g(f(\pi))$ and $S(\pi)=R(f(\pi))$, and the solution of the maximization problem (2.12). The latter can be obtained by numerically computing the integral in (2.12) for different vectors α, so as to find out the vector $\overline{\alpha}$ which maximizes the expectation of the net toll revenues for an assigned probability law of parameter β.

The functions $y(\pi)$ and $S(\pi)$ can be computed by solving problem (2.1) or (2.2) for a wide range of values of π, and calculating the corresponding values assumed by the two functions. Calculating the regressions of these values on the corresponding π leads to a polynomial approximations of the relations between each y and S, and π.

4. An Example of the Application of the Model

The model has been applied as a study case to the coastal road network between Pisa and Grosseto in Tuscany, which includes a four-lane divided freeway henceforth referred to as road A. A long-standing debate is under way regarding the conversion of road A into a motorway, involving substantial modernization and the institution of a toll. The graph representing the network is made up of 356 links and 112 nodes, of which 29 are origin-destination centroids of transportation demand, one for each township in the area.

This study case regards the levying of a toll on road A during the peak period of the day. As a measure of the damage consequent to institution of the toll, we have considered a quantity equal to twice the sum of the increases in travel time (with respect to the situation in the absence of toll) of a vehicle on the links alternative to road A in the network under examination.

In calculating the equilibrium flows in the network, it has been assumed that the costs to motorists to move from one extremity to the other of each link of the graph is a vector made up of two components: the monetary cost c_m and the travel time t. The monetary cost perceived on the average by motorists has been estimated at 0.15 EUROs/km, while the time has been expressed as a function of the traffic flow f in that link, $t = t(f)$, assuming a different functional relation for the various types of roads in the network.

The vector cost has been converted into a scalar quantity measured in monetary units by multiplying the time t by a coefficient v_t, which measures the monetary value of a unit of time perceived by motorists. The costs function associated to each link therefore takes on the expression:

$$c(f) = c_m + v_t t(f) \tag{4.13}$$

In order to account for the fact that the monetary value of time as perceived by motorists varies randomly from one individual to the next, coefficient v_t has been considered to be a random variable normally distributed around a central value, VT = 0.05 EUROs/min, with variance $\sigma^2 = 0,09 \cdot VT^2$.

Calculation of the equilibrium flows has therefore been tackled via a stochastic procedure, by solving problem (2.2). In order to calculate expressions (2.6) and (2.7) of functions $y(\pi)$ and $S(\pi)$, 16 different values of π, ranging from 0 to 0.15 EURO/km, have been assigned to road A, and the values of the damage due to toll, y (min/h), and of toll revenues, S (EURO/h), have thus been calculated for each value of the toll, π. The expressions for functions $y(\pi)$ and $S(\pi)$ have been obtained by executing a regression of the values of y and S onto the corresponding π. They are furnished in the following, together with their respective statistics R^2:

$$y = 984.84\pi + 354.94\pi^2 \qquad R^2 = 0,9999 \tag{4.14}$$

$$S = 232506.32\pi - 1413627.44\pi^2 \qquad R^2 = 0,9999 \tag{4.15}$$

The values of the statistics R^2 underscore the high degree of approximation with which the regression equations reproduce the real functional relations, which are quadratic functions, strictly convex and concave respectively, as we said before.

Thus the expression (2.10) of $\pi(\alpha, \beta)$ is written as follows:

$$\pi(\alpha,\beta)=\frac{\alpha_1-984.84\beta}{2(\alpha_2+354.94\beta)} \quad \text{if } \pi(\alpha,\beta)>0, \text{ otherwise } \pi(\alpha,\beta)=0 \tag{4.16}$$

Let us now suppose that the public agency assigns β a normal probability law with mean μ_β and variance $\sigma_\beta^2=(\eta\mu_\beta)^2$. Designating $f_\beta(\beta)$ as the probability density function of β, the expected value of net toll revenues for given values of α, μ_β, η is:

$$V(\alpha,\mu_\beta,\eta)=\int_{\beta_1}^{\beta_2}[S(\pi(\alpha,\beta))-\xi(\alpha,\pi(\alpha,\beta))]f_\beta(\beta)\mathrm{d}\beta \tag{4.17}$$

In (4.17) the extrema, β_1 and β_2, of the definition interval of $f_\beta(\beta)$, needed for numerical calculation of the integral, have been assumed to correspond to values of 0.0005 and 0.9995 of the distribution function of β, while the value of $\pi(\alpha\beta)$ for each β is furnished by (4.16).

By repeating the calculation of $V(\alpha\mu_\beta,\eta)$, with given μ_β and η, for a wide range of values of α_1 and α_2, we build the image of V, and therefore can determine the vector $\overline{\alpha}(\mu_\beta,\eta)$ that maximizes V, and from (4.16) the corresponding toll chosen by the local communities that corresponds to the specific value they have attributed to parameter β.

The calculations have been repeated for many values of μ_β and η. The results obtained have confirmed that, if the value attributed to β by the local communities is quite different from the central value μ_β chosen by the public agency, the loss in net revenues is acceptable if the latter assumes that the variance of β is large, while the loss is very great if the variance of β is supposed to be low.

Consider for instance the case in which $\mu_\beta=$ 15 EURO/min. The compensation functions computed by solving problem (2.12) with two, quite different, values of η, give rise to the following net toll revenues (EURO/h) in correspondence of three values of β chosen by the local communities:

η	$\beta=9$	$\beta=15$	$\beta=21$	
0,10	4942	7998	4436	(4.18)
0,25	7228	7720	6370	(4.19)

If the public agency knew the values attributed to β by the local communities, the compensation function computed by solving problem (2.5) would give rise to the following net toll revenues:

$$\begin{array}{ccc} \beta = 9 & \beta = 15 & \beta = 21 \\ 8304 & 8788 & 7837 \end{array} \tag{4.20}$$

The comparison of results (4.20) with the corresponding (4.18) and (4.19) puts in evidence the losses in net toll revenues due to the fact that the public agency does not know the value attributed to β by the local communities. These losses are still acceptable, even if this β value is quite different from μ_β, in case the public agency assumes a rather great value for η, whereas they are very high, to the point of rendering the institution of toll uneconomical, if the value attributed to η is too little.

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